Lowest eigenvalues of the Dirac operator for two color QCD at nonzero chemical potential

Elmar Bittner[a], Maria-Paola Lombardo[b], Harald Markum[a], and Rainer Pullirsch[a]

[a]Atominstitut, Technische Universität Wien, A-1040 Vienna, Austria

[b]Istituto Nazionale di Fisica Nucleare, Sezione di Padova, e Gruppo Collegato di Trento, Italy

We investigate the eigenvalue spectrum of the staggered Dirac matrix in SU(3) and U(1) gauge theory as well as in full QCD with two colors and finite chemical potential. Along the strong-coupling axis up to the phase transition, the low-lying Dirac spectrum of these quantum field theories is well described by random matrix theory and exhibits universal behavior. Related results for gauge theories with minimal coupling are discussed in the chirally symmetric phase and no universality is seen for the microscopic spectral densities.

1. Spectrum in confinement

The eigenvalues of the Dirac operator are of great interest for the universality of important features of QCD and QED. The accumulation of small eigenvalues is, via the Banks-Casher formula [1], related to the spontaneous breaking of chiral symmetry. Their properties are known to be described by random matrix theory (RMT) in the confinement, see Ref. [2].

The Dirac operator itself, $\not{D} = \not{\partial} + ig\not{A}$, is anti-Hermitian so that the eigenvalues λ_n of $i\not{D}$ are real. Because of $\{\not{D}, \gamma_5\} = 0$ the nonzero λ_n occur in pairs of opposite sign. In the presence of a chemical potential $\mu > 0$, the Euclidean action becomes a complex number and the fermionic matrix becomes a non-Hermitian matrix.

2. Spectrum into deconfinement

We have continued our investigations [3] with a study of the distribution of the small eigenvalues in the whole phase diagram. The Banks-Casher formula [1] relates the Dirac eigenvalue density $\rho(\lambda)$ at $\lambda = 0$ to the chiral condensate, $\Sigma \equiv |\langle\bar{\psi}\psi\rangle| = \lim_{\varepsilon\to 0}\lim_{V\to\infty}\pi\rho(\varepsilon)/V$. The microscopic spectral density, $\rho_s(z) = \lim_{V\to\infty}\rho\left(z/V\Sigma\right)/V\Sigma$, should be given by the appropriate prediction of RMT [4]. In the case of complex eigenvalues the situation is more complicated [5] and $\rho(0)$ can be used as a lower bound for Σ.

We present results for the density of the small eigenvalues in Fig. 1 for SU(3) theory and the staggered Dirac operator on a 4^4 lattice from 5000 configurations for $\beta = 5.4$ and 3000 configurations for $\beta = 5.6$ and $\beta = 5.8$ around the critical temperature $\beta_c \approx 5.7$. Our analogous presentation for U(1) theory is from 10000 configurations on a 4^4 lattice around the critical coupling $\beta_c \approx 1.01$. In the confinement phase it was shown that, both the microscopic spectral density $\rho_s(z)$ and the distribution $P(\lambda_{\min})$ of the smallest eigenvalue agree with the RMT predictions of the chiral unitary ensemble for topological charge $\nu = 0$ [6,2]. In the case of two-color QCD with staggered fermions on a 6^4 lattice we produced at least 2100 configurations for each value of μ around the critical chemical potential $\mu_c \approx 0.3$ keeping $\beta = 1.3$ fixed [7]. Since the eigenvalues move into the complex plane for $\mu > 0$, a band of width $\epsilon = 0.015$ parallel to the imaginary axis was considered to construct $\rho(y)$, i.e. $\rho(y) \equiv \int_{-\epsilon}^{\epsilon} dx\, \rho(x,y)$, where $\rho(x,y)$ is the density of the complex eigenvalues $x + iy$. Alternatively $\rho(|\lambda|)$ was constructed from the absolute value for $|\lambda|$ small and is presented in this write-up.

The distribution of the lowest eigenvalue is displayed in Fig. 2 for the theories under investigation. The data of the SU(3) and U(1) theories in the deconfinement could not be fitted to the width of the functional form derived for RMT in

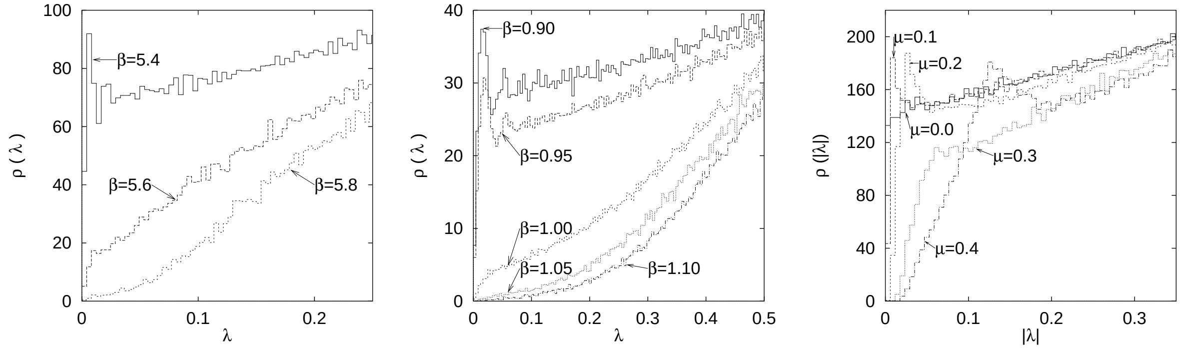

Figure 1. Density $\rho(\lambda)$ of small eigenvalues for SU(3) (left) and U(1) gauge theory (center) as well as for two-color QCD (right) across the transition of critical coupling and critical chemical potential, respectively.

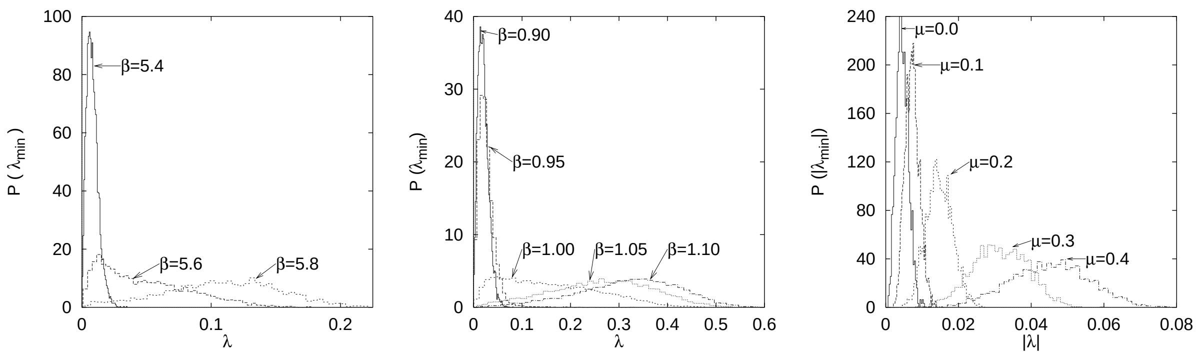

Figure 2. Distribution $P(\lambda_{\rm min})$ of the smallest eigenvalue as in Fig. 1.

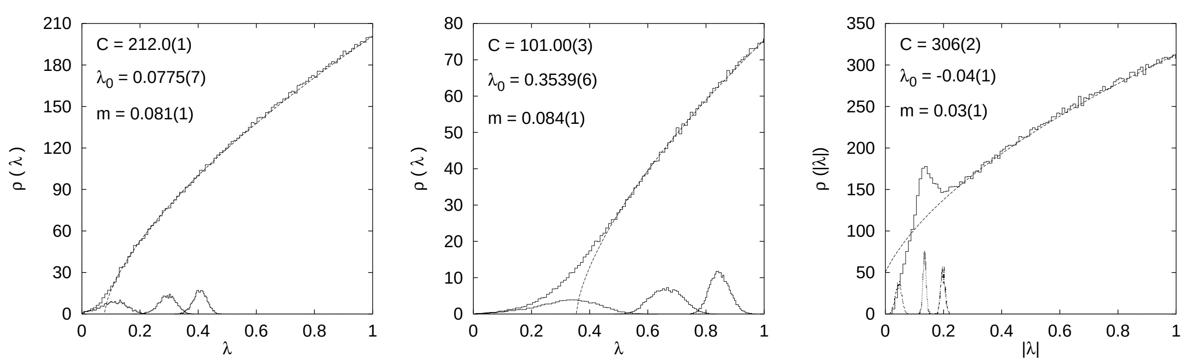

Figure 3. Fit of the spectral density to $\rho(\lambda) = C(\lambda - \lambda_0)^{2m+1/2}$ in SU(3) at $\beta = 5.8$ (left), in U(1) at $\beta = 1.10$ (center) and in two-color QCD at $\mu = 0.4$ (right). The contribution of the smallest eigenvalue, the 11$^{\rm th}$ eigenvalue and the 21$^{\rm st}$ eigenvalue is inserted.

the chirally broken phase. The quality of the data of two-color QCD was not sufficient for a reliable fit to a trial function for $P(|\lambda_{\rm min}|)$ except for the RMT prediction of the chiral symplectic ensemble at $\mu = 0$. The lattice results of the three theories look very similar and might give some help for a derivation within RMT.

Nevertheless, the quasi-zero modes which are responsible for the chiral condensate $\Sigma \neq 0$ build up when we cross from the deconfinement into the confined phase. Figures 1 and 2 demonstrate that both $\rho(\lambda)$ and $P(\lambda_{\rm min})$ plotted with varying β or μ on identical scales, respectively, can serve as an indicator for the phase transition. The change in bending from positive to negative curvature of $\rho(0)$ as a function of coupling/density might serve to pin down the critical point.

In Fig. 3 we turn to a discussion of the spectrum in the quark-gluon plasma and Coulomb phase. From RMT a functional form of $\rho(\lambda) = C(\lambda - \lambda_0)^{2m+1/2}$ is expected at the onset of the eigenvalue density [8]. A fit to the data in the regime up to $\lambda = 1$ yields $m = 0.081(1)$ for SU(3) and $m = 0.084(1)$ for the Abelian theory, in agreement with recent studies [9]. This suggests that both theories correspond to universality class $m = 0$. For this class a microscopic level density involving the Airy function can be deduced from RMT [10]. A rescaling of our data from the 4^4 lattice to this functional form is not satisfactory for both theories [9]. We tried a fit for two-color QCD assuming the above functional form for $\rho(|\lambda|)$ in a region up to $|\lambda| = 1$ obtaining $m = 0.03(1)$ consistent with universality class $m = 0$.

3. Conclusion

We investigated universality concerning the low-lying spectra of the Dirac operators of both SU(3) and U(1) gauge theories. One finds that in the phase in which chiral symmetry is spontaneously broken the distribution $P(\lambda_{\rm min})$ and the microscopic spectral density $\rho_s(z)$ are described by chiral RMT. For two-color QCD with chemical potential we were not able to verify such a relation for $0 < \mu < \mu_c$, not only because of our data but also in lack of an analytic result for non-Hermitian RMT [11]. In the phase where chiral symmetry is restored one has to rely on ordinary RMT. Here we find universal behavior only of the macroscopic density $\rho(\lambda)$ for all gauge theories with minimal coupling.

Acknowledgments: This study was supported in part by FWF project P14435-TPH.

REFERENCES

1. T. Banks and A. Casher, Nucl. Phys. B 169 (1980) 103.
2. M.E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J.J.M. Verbaarschot, and T. Wettig, Phys. Rev. Lett. 80 (1998) 1146; B.A. Berg, H. Markum, R. Pullirsch, and T. Wettig, Phys. Rev. D 63 (2001) 014504.
3. E. Bittner, H. Markum, and R. Pullirsch, Nucl. Phys. B (Proc. Suppl.) 96 (2001) 189; E. Bittner, M.-P. Lombardo, H. Markum, and R. Pullirsch, Nucl. Phys. B (Proc. Suppl.) 94 (2001) 445.
4. E.V. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. A 560 (1992) 306; J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. 70 (1993) 3852; J.J.M. Verbaarschot, Phys. Rev. Lett. 72 (1994) 2531.
5. K. Splittorff, D. Toublan, and J.J.M. Verbaarschot, hep-ph/0108040.
6. M. Göckeler, H. Hehl, P.E.L. Rakow, A. Schäfer, and T. Wettig, Phys. Rev. D 59 (1999) 094503.
7. S. Hands, J.B. Kogut, M.-P. Lombardo, and S.E. Morrison, Nucl. Phys. B 558 (1999) 327.
8. M.J. Bowick and E. Brézin, Phys. Lett. B 268 (1991) 21; E. Kanzieper and V. Freilikher, Phys. Rev. E 55 (1997) 3712.
9. F. Farchioni, P. de Forcrand, I. Hip, C.B. Lang, and K. Splittorff, Phys. Rev. D 62 (2000) 014503; P.H. Damgaard, U.M. Heller, R. Niclasen, and K. Rummukainen, Nucl. Phys. B 583 (2000) 347.
10. P.J. Forrester, Nucl. Phys. B 402 (1993) 709.
11. G. Akemann, hep-th/0106053.